Invariant shape similarity
Invariant similarity

\[ d(\tau X, \sigma Y) = d(X, Y) \]
Equivalence

$X = Y$

Equal

$\exists i \in \text{Iso}(E) \text{ s.t. } i(X) = Y$

Congruent

$\exists \varphi : X \to Y \text{ s.t. } d_Y \circ (\varphi \times \varphi) = d_X$

Isometric
Equivalence

Equivalence is a **binary relation** $\sim$ on the **space of shapes** $\mathcal{S}$ which for all $X, Y, Z \in \mathcal{S}$ satisfies

- **Reflexivity**: $X \sim X$
- **Symmetry**: $X \sim Y \Rightarrow Y \sim X$
- **Transitivity**: $X \sim Y$ and $Y \sim Z \Rightarrow X \sim Z$

Can be expressed as a **binary function** $d : \mathcal{S} \times \mathcal{S} \rightarrow \{0, 1\}$

$$d(X, Y) = 0 \text{ if and only if } X \sim Y$$

**Quotient space** $\mathcal{S}^* = \mathcal{S} \setminus \sim$ is the space of **equivalence classes**
Equivalence
Equivalence

All deformations of the human shape are “the same”
Similarity

- Shapes are rarely truly equivalent (e.g., due to acquisition noise or since most shapes are rigid)
- We want to account for “almost equivalence” or similarity
- $\varepsilon$-similar $= \varepsilon$-isometric (w.r.t. some metric)
- Define a distance on the shape space $S$ quantifying the degree of dissimilarity of shapes
Similarity

A monkey shape is more similar to a deformation of a monkey shape…

…than to a human shape
Isometry-invariant distance

Non-negative function $d : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ satisfying for all $X, Y, Z \in \mathcal{S}$

- **Similarity:** $d(X, Y) \leq \epsilon \Rightarrow X$ and $Y$ are $c\epsilon$-isometric;

  $X$ and $Y$ are $\epsilon$-isometric $\Rightarrow d(X, Y) \leq c\epsilon$

  (In particular, $d(X, Y) = 0$ if and only if $X \sim Y$)

- **Symmetry:** $d(X, Y) = d(Y, X)$

- **Triangle inequality:** $d(X, Y) + d(Y, Z) \geq d(X, Z)$

  **Corollary:** $d$ is a metric on the quotient space $\mathcal{S}^* = \mathcal{S}/\sim$

Given discretized shapes $X^r_N = \{x_1, \ldots, x_N\}$ and $Y^r_M = \{y_1, \ldots, y_M\}$ sampled with radius $r$

- **Consistency to sampling:** $\lim_{r \to 0} \hat{d}(X^r_N, Y^r_M) = d(X, Y)$
Canonical forms distance

Minimum-distortion embedding

\[ f = \arg \min_{f:X \to \mathbb{Z}} \text{dis}_f \quad (\mathbb{Z}, d_{\mathbb{Z}}) \]

Minimum-distortion embedding

\[ g = \arg \min_{g:Y \to \mathbb{Z}} \text{dis}_g \]

Compute Hausdorff distance over all isometries in \((\mathbb{Z}, d_{\mathbb{Z}})\)

\[ d_{\text{CF}}(X, Y) = \min_{f:X \to \mathbb{Z}} \min_{g:Y \to \mathbb{Z}} d_{\mathbb{H}}(f(X), g(Y)) \]

No fixed embedding space \(\mathbb{Z}\) will give \textbf{distortion-less} canonical forms
Invariant shape similarity

Gromov-Hausdorff distance

Isometric embedding

\( d_{GH}(X, Y) = \inf_{f, g} d_H^Z(f(X), g(Y)) \)

\( f: X \rightarrow Z \)
\( g: Y \rightarrow Z \)

Gromov-Hausdorff distance: include \( Z \) into minimization

Mikhail Gromov
Properties of Gromov-Hausdorff distance

- **Metric** on the quotient space $S^* = S \setminus \sim$ of isometries of shapes

- **Similarity:** $d_{GH}(X, Y) \leq \epsilon \Rightarrow X$ and $Y$ are $2\epsilon$-isometric;

  $X$ and $Y$ are $\epsilon$-isometric $\Rightarrow d_{GH}(X, Y) \leq 2\epsilon$

- **Consistent to sampling:** given discretized shapes $X^r_N = \{x_1, \ldots, x_N\}$ and $Y^r_M = \{y_1, \ldots, y_M\}$ sampled with radius $r$

  $$|d_{GH}(X, Y) - d_{GH}(X^r_N, Y^r_M)| \leq 2r$$

- **Generalization of Hausdorff distance:**
  - **Hausdorff distance** between subsets of a metric space
  - **Gromov-Hausdorff distance** between metric spaces

Gromov, 1981
Alternative definition I (metric coupling)

\[ d_{GH}(X, Y) = \inf_{d_Z} d_{H,(X \sqcup Y, d_Z)}(X, Y) \]

where

- \( Z = X \sqcup Y \) is the disjoint union of \( X \) and \( Y \)
- the (semi-) metric \( d_Z \) satisfies \( d_Z|_{X \times X} = d_X \) and \( d_Z|_{Y \times Y} = d_Y \)
Alternative definition I (metric coupling)

\[ d_{GH}(X, Y) = \inf_{d_Z} d_H^{(X \sqcup Y, d_Z)}(X, Y) \]

\[
d_Z = \begin{array}{ccc}
X & & Y \\
& d_X & D \\
D & D & d_Y \\
& Y &
\end{array}
\]

Optimization over \( d_Z \) translates into finding the values of \( D \)

\[ d_{GH}(X, Y) = \inf_D \max\{\sup_x \inf_y D(x, y), \sup_y \inf_x D(x, y)\} \]

s.t. \( D(x, y) \leq D(x', y) + D(x, x') \forall x, x' \in X, y \in Y \)

A lot of constraints!

Mémoli, 2008
Correspondence

A subset $C \subset X \times Y$ is called a correspondence between $X$ and $Y$ if for every $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in C$ and similarly for every $y \in Y$ there exists $x \in X$ such that $(x, y) \in C$.

Particular case: given $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$

$$C = (X \times \varphi(X)) \cup (\psi(Y) \times Y)$$
Correspondence distortion

The distortion of correspondence $C$ is defined as

$$\text{dis} C = \sup_{(x,y) \in C, (x',y') \in C} |d_X(x, x') - d_Y(y, y')|$$
Alternative definition II (correspondence distortion)

\[ d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathcal{C}} \text{dis} \mathcal{C} \]

Proof sketch

1. Show that for any \( r > d_{GH}(X, Y) \) there exists \( \mathcal{C} \) with \( \text{dis} \mathcal{C} < 2r \)

Since \( d_{GH}(X, Y) < r \), by definition of \( d_{GH} \), \( X \) and \( Y \) are subspaces of some \( (\mathbb{Z}, d_{\mathbb{Z}}) \) such that \( d_{H}(X, Y) < r \)

Let \( \mathcal{C} = \{(x, y) : x \in X, y \in Y, d_{\mathbb{Z}}(x, y) < r\} \)

By triangle inequality, for \((x, y), (x', y') \in \mathcal{C}\)

\[ d_{\mathbb{Z}}(x, x') \leq d_{\mathbb{Z}}(x, y) + d_{\mathbb{Z}}(x', y') \leq d_{\mathbb{Z}}(x, y) + d_{\mathbb{Z}}(x', y') + d_{\mathbb{Z}}(y, y') \]

\[ -d_{\mathbb{Z}}(d_{\mathbb{Z}}(y), x') \leq d_{\mathbb{Z}}(x, y) + d_{\mathbb{Z}}(x', y') - d_{\mathbb{Z}}(x, y) - d_{\mathbb{Z}}(x', y') + d_{\mathbb{Z}}(y, y') \]
Alternative definition II (correspondence distortion)

2. Show that \( d_{GH}(X, Y) < \frac{1}{2} \text{dis} \mathcal{C} \) for any \( \mathcal{C} \)

Let \( \text{dis} \mathcal{C} = 2r \)

It is sufficient to show that there is a (semi-)metric \( d_{\mathbb{Z}} \) on the disjoint union \( \mathbb{Z} = X \sqcup Y \) such that \( d_{\mathbb{Z}}|_{X \times X} = d_X, d_{\mathbb{Z}}|_{Y \times Y} = d_Y \), and \( d_{\mathbb{Z}}^H(X, Y) \leq r \)

Construct the metric \( d_{\mathbb{Z}}|_{X \times Y} \) as follows

\[
d_{\mathbb{Z}}(x, y) = \inf_{(x', y') \in \mathcal{C}} \{d_X(x, x') + d_Y(y, y')\} + r
\]

(in particular, \( d_{\mathbb{Z}}(x, y) = r \) for \( (x, y) \in \mathcal{C} \)).
Alternative definition II (correspondence distortion)

First, \( d^Z_H(X, Y) = \max\{\sup_x \inf_y d_Z(x, y), \sup_y \inf_x d_Z(x, y)\} \)

\[ = r \]

For each \( x \), \( \exists y : (x, y) \in C \)

Since \( d_Z(x, y) = r \) for \( (x, y) \in C \),

\[ \inf_y d_Z(x, y) = r \]

Second, we need to show that \( d_Z \) is a (semi-)metric on \( Z = X \sqcup Y \)

On \( X \times X \) and \( Y \times Y \), it is straightforward

We only need to show metric properties hold on \( X \times Y \)
Alternative definition III

\[ d_{GH}(X, Y) = \frac{1}{2} \inf_{\varphi: X \rightarrow Y, \psi: Y \rightarrow X} \max \{ \text{dis} \varphi, \text{dis} \psi, \text{dis}(\varphi, \psi) \} \]

measures how much \( d_X \) is distorted by \( \varphi \) when embedded into \( d_Y \)
Alternative definition III

\[ d_{GH}(X, Y) = \frac{1}{2} \inf_{\varphi: X \to Y, \psi: Y \to X} \max\{\text{dis} \varphi, \text{dis} \psi, \text{dis}(\varphi, \psi)\} \]

measures how much \( d_Y \) is distorted by \( \psi \) when embedded into \( d_X \)
**Alternative definition III**

\[ d_{GH}(X, Y) = \frac{1}{2} \inf_{\varphi:X \rightarrow Y, \psi:Y \rightarrow X} \max\{\text{dis} \varphi, \text{dis} \psi, \text{dis}(\varphi, \psi)\} \]

measures how far is \( \varphi \) from being the inverse of \( \psi \)
Generalized MDS

\[
\min_{\{y_1, \ldots, y_n\} \subseteq Y} \left\| d_X(x_i, x_j) - d_Y(y_i, y_j) \right\|
\]

A. Bronstein, M. Bronstein & R. Kimmel, 2006
Difficulties

\[ \min_{\{y_1, \ldots, y_n\} \subset Y} \| d_X(x_i, x_j) - d_Y(y_i, y_j) \| \]

- How to represent points on \( Y \)?
  - **Global** parametrization is not always available.
  - Some **local** representation is required in general case.

- No more closed-form expression for \( d_Y \).
  - Metric needs to be approximated.

- Minimization algorithm.
Local representation

- $Y$ is sampled at $Y_M = \{y_1, \ldots, y_M\}$ and represented as a triangular mesh $T(Y_M)$.
- Any point $y'_i \in T(Y_M)$ falls into one of the triangles $t_i \in T$.
- Within the triangle, it can be represented as convex combination of triangle vertices $x_{t_i,1}, x_{t_i,2}, x_{t_i,3}$,

$$y'_i = u_i^1 x_{t_i,1} + u_i^2 x_{t_i,2} + u_i^3 x_{t_i,3}$$
$$= u_i^1 x_{t_i,1} + u_i^2 x_{t_i,2} + (1 - u_i^1 - u_i^2) x_{t_i,3}$$

- Barycentric coordinates $y'_i = (t_i, u_i)$.
- We will need to handle discrete indices $t_i$ in minimization algorithm.
Geodesic distances

- Distance terms $d_X(x_i, x_j)$ can be precomputed, since $\{x_i\}$ are fixed.
- How to compute distance terms $d_Y(y'_i, y'_j)$?
- No more closed-form expression.
- Cannot be precomputed, since $\{y'_i\}$ are minimization variables. $y'_i$ can fall anywhere on the mesh.

- Precompute $(D_Y)_{ij} = d_Y(y_i, y_j)$ for all $i, j = 1, ..., M$.
- Approximate
  $$\tilde{d}_Y(y'_i, y'_j) \approx d_Y(y'_i, y'_j)$$
  for any $y'_i, y'_j \in T(Y_M)$. 

Geodesic distance approximation

- Approximation $\hat{d}_Y(y'_i, y'_j) \approx d_Y(y'_i, y'_j)$ from $D_Y$.

- First order accurate: $|\hat{d}_Y(y'_i, y'_j) - d_Y(y'_j, y'_i)| = O(r)$

- Consistent with data: $\hat{d}_Y(y_i, y_j) = d_Y(y_i, y_j)$

- Symmetric: $\hat{d}_Y(y'_i, y'_j) \approx \hat{d}_Y(y'_j, y'_i)$

- Smoothness: $\hat{d}_Y(y'_i, y'_j)$ is $C^1$ and a closed-form expression for its derivatives is available to minimization algorithm. Might be only $C^0$ at some points or along some lines.

- Efficiently computed: constant complexity independent of $M$. 
Geodesic distance approximation

- Compute $\tilde{d}_Y(y, y')$ for $y = (t, u), y' = (t', u') \in T(Y_M)$.
- $y$ falls into triangle $t = (y_1, y_2, y_3)$ and is represented as
  \[
  y = u^1 y_1 + u^2 y_2 + u^3 y_3 = u^1 y_1 + u^2 y_2 + (1 - u^1 - u^2) y_3
  \]
- Particular case: $y' \in Y_M$
- Hence, we can precompute distances
  \[
  d_1 = d_Y(y_1, y')
  d_2 = d_Y(y_2, y')
  d_3 = d_Y(y_3, y')
  \]
- How to compute $\tilde{d} = \tilde{d}_Y(y, y')$ from $d_i$?
Geodesic distance approximation

- We have already encountered this problem in fast marching.
- **Wavefront** arrives at triangle vertex $y_i$ at time $d_i$.
- When does it arrive to $y$?
- Adopt **planar wavefront model**.
- Distance map is **linear** in the triangle (hence, linear in $u$)
- Solve for coefficients and obtain a linear interpolant

$$
\bar{d}_Y(y, y') = u^1 d_1 + u^2 d_2 + u^3 d_3 = d^T u
$$
Geodesic distance approximation

- **General case**: $y'$ falls into triangle $t' = (y_4, y_5, y_6)$ and is represented as
  \[
  y' = u'^1 y_4 + u'^1 y_5 + u'^3 y_6 \\
  = u'^1 y_4 + u'^2 y_5 + (1 - u'^1 - u'^2) y_6
  \]

- Apply previous steps in triangle $t$ to obtain
  \[
  \begin{pmatrix}
  \hat{d}_4 \\
  \hat{d}_5 \\
  \hat{d}_6
  \end{pmatrix} = \begin{pmatrix}
  \hat{d}_Y(y, y_4) \\
  \hat{d}_Y(y, y_5) \\
  \hat{d}_Y(y, y_6)
  \end{pmatrix} = \begin{pmatrix}
  d_{14} & d_{24} & d_{34} \\
  d_{15} & d_{25} & d_{35} \\
  d_{16} & d_{26} & d_{36}
  \end{pmatrix} u = D_Y(t, t') u
  \]

- Apply once again in triangle $t'$ to obtain
  \[
  \hat{d}_Y(y, y') = u'^T D_Y(t, t') u
  \]
Un ballo a quattro passi

Step 1: $\hat{d}_4 \approx d_Y(y, y_4)$ is computed from $d_{14}, d_{24}, d_{34}$ in the triangle $y_1, y_2, y_3$.

Step 2: $\hat{d}_5 \approx d_Y(y, y_5)$ is computed from $d_{15}, d_{25}, d_{35}$ in the triangle $y_1, y_2, y_3$.

Step 3: $\hat{d}_6 \approx d_Y(y, y_6)$ is computed from $d_{16}, d_{26}, d_{36}$ in the triangle $y_1, y_2, y_3$.

Step 4: $\hat{d} \approx d_Y(y, y')$ is computed from $\hat{d}_4, \hat{d}_5, \hat{d}_6$ in the triangle $y_4, y_5, y_6$. 
Minimization algorithm

- How to minimize the generalized stress?
- **Particular case:** $L_2$ stress

\[
\sigma(t_1, u_1, \ldots, t_N, u_N) = \sum_{j>i} \left( d_X(x_i, x_j) - u_i^T D_Y(t_i, t_j) u_j \right)^2
\]

- Fix all $u_j$ and all $t_j$ except for some $u_i$.
- Stress as a function of $u_i$ only becomes **quadratic**

\[
\sigma(u_i) = \sum_{j>i} \left( d_X(x_i, x_j) - u_i^T D_Y(t_i, t_j) u_j \right)^2
\]
Quadratic stress

- Quadratic stress \( \sigma(u_i) = u_i^T A_i u_i + 2b_i^T u_i + c_i \)

\[
A_i = \sum_{j \neq i} D_Y(t_i, t_j) u_j u_j^T D_Y(t_i, t_j)^T \\
b_i = -\sum_{j \neq i} d_X(x_i, x_j) D_Y(t_i, t_j) u_j \\
c_i = \sum_{j \neq i} d_X^2(x_i, x_j)
\]

- \( A_i \) is **positive semi-definite**.
- \( \sigma \) is **convex** in \( u_i \) (but not necessarily in \( u_1, \ldots, u_N \) together).
Quadratic stress

- Closed-form solution for minimizer of $\sigma(u_i)$

$$u_i^* = \arg\min_{u_i} \sigma(u_i) = \arg\min_{u_i} u_i^T A_i u + 2b_i^T u + c_i = -A_i^{-1}b_i$$

- Problem: solution might be outside the triangle.
- Solution: find constrained minimizer

$$u_i^* = \arg\min_{u_i} \sigma(u_i) \text{ s.t. } \begin{cases} u_i \geq 0 \\ u_i^1 + u_i^2 + u_i^3 = 1 \end{cases}$$

- Closed-form solution still exists.
Minimization algorithm

- Initialize \( \{u_i, t_i\} \)
- For each \( i = 1, \ldots, N \)
  - Fix \( u_j \neq i, t_j \) and compute gradient
    \[
    g_i = \nabla \sigma(u_i) = 2A_i u_i + 2b_i
    \]
- Select \( i \) corresponding to maximum \( \|g_i\| \).
- Compute minimizer
  \[
  u_i = \arg \min_{u_i \geq 0} \sigma(u_i) \quad \text{s.t.} \quad u_i^1 + u_i^2 \leq 1
  \]
- If constraints are active
  translate \( u_i, t_i \) to adjacent triangle.
- Iterate until convergence…
How to move to adjacent triangles?

- Three cases
  - All $u_i > 0$: $u_i$ inside triangle.
  - $u_i = 0$: $u_i$ on edge opposite to $x_i$.
  - $u_i = 1$: $u_i$ on vertex $x_i$. 

(0.2, 0.3, 0.5) inside
(0, 0.4, 0.6) on edge
(0, 0, 1) on vertex
**Point on edge**

- $u$ on edge opposite to $x_i$.

- If edge is **not shared** by any other triangle, we are on the **boundary** – no translation.

- Otherwise, express the point as $u'$ in triangle $t'$.
  - $u'$ contains same values as $u$.
  - May be **permuted** due to different vertex ordering in $t'$.

- *Complication:* $\sigma$ is not $C^1$ on the edge.
  - Evaluate **gradient** $\nabla \sigma(u')$ in $t'$.
  - If $-\nabla \sigma(u')$ points **inside triangle**, update $(t, u)$ to $(t', u')$.
Point on vertex

- \( u \) on vertex \( x_i \).
- For each triangle \( t' \) sharing vertex \( x_i \):
  - Express point as \( u' \) in \( t' \).
  - Evaluate gradient \( \nabla \sigma(u') \) in \( t' \).
- Reject triangles with \( -\nabla \sigma(u') \) pointing outside.
- Select triangle \( t' \) with maximum \( \| \nabla \sigma(u') \| \).
- Update \((t, u)\) to \((t', u')\).
MDS vs GMDS

MDS

- Stress
  \[ \| d_X(x_i, x_j) - d_{\mathbb{R}^n}(z_i, z_j) \| \]

- Analytic expression for \( d_{\mathbb{R}^n} \)
- Nonconvex problem
- Variables: Euclidean coordinates of the points
  \( \{x_1, \ldots, x_n\} \subset \mathbb{R}^n \)

Generalized MDS

- Generalized stress
  \[ \| d_X(x_i, x_j) - d_Y(y_i, y_j) \| \]

- \( d_Y \) must be interpolated
- Nonconvex problem
- Variables: points on \( Y \) in barycentric coordinates
  \( \{y_1, \ldots, y_n\} \)
Multiresolution

- Stress is non convex – many small local minima.
- Straightforward minimization gives poor results.
- How to initialize GMDS?

Multiresolution:

- Create a hierarchy of grids in $\mathbb{X}$,
  
  $$\Omega_L \subset \Omega_{L-1} \subset \cdots \subset \Omega_0$$

- Each grid comprises
  
  - Sampling: $X_l = \{x_1, \ldots, x_{N_l}\} \subseteq T(X_N)$
  - Geodesic distance matrix: $D_l$
    
    $$(D_l)_{ij} = d_X(x_i, x_j)$$
Multiresolution

- **Initialize** $Y_L$ at the coarsest resolution in $Y$.

- For $l = L, L-1, ..., 0$
  - Starting at initialization $Y_l$, solve the **GMDS problem**
    
    $$Y_l^* = \arg \min_{\{y'_1, ..., y'_{N_l} \in Y\}} \sigma(y'_1, ..., y'_{N_l}; D_l)$$

  - **Interpolate** solution to next resolution level
    $$Y_{l-1} = P_{l-1}^l Y_l^*$$

- Return $Y_0^*$.
Numerical Geometry of Non-Rigid Shapes

Invariant shape similarity

GMDS

Interpolation

$\Omega_1$

$\Omega_0$
Multiresolution encore

So far, we created a hierarchy of **embedded spaces** $X_1, \ldots, X_L$.

One step further: create a hierarchy of **embedding spaces** $Y_1, \ldots, Y_L$. 
Labeling problem

- Build a graph with vertices $V = X \times Y$ and edges $E = V \times V$
- Label each vertex
  
  $u_{xy} = \begin{cases} 
  1 & (x, y) \in C \\
  0 & \text{else} 
  \end{cases}$

- Minimum distortion correspondence = graph labeling problem

  \[
  \min_{u \in \{0,1\}} \sum_{((x,y),(x',y')) \in E} u_{xy} u_{x'y'} |d_X(x, x') - d_Y(y, y')|^2 
  \]

  s.t. \( \sum_{x \in X} u_{xy} \geq 1, \forall y \in Y; \sum_{y \in X} u_{xy} \geq 1, \forall x \in X \)

- Efficient solvers with good global convergence properties
- Complexity: $O(|V|^2|E|) = O(N^8)$
- Hierarchical solution complexity can be lowered to $O(N^4)$

Torresani, Kolmogorov, Rother 2008
Wang, B 2010
MATLAB® intermezzo
GMDS
**Discrete Gromov-Hausdorff distance**

- Two coupled GMDS problems

\[ d_{GH}(X_N, Y_M) = \frac{1}{2} \min_{y'_1, \ldots, y'_N \in Y, x'_1, \ldots, x'_M \in X} \max \left\{ \frac{1}{2} \sum_{i,j} \frac{1}{|X|} \left( |d_X(x_i, x_j) - d_Y(y'_i, y'_j)|, \right) \right\} \]

- Can be cast as a *constrained problem*

\[ d_{GH}(X_N, Y_M) = \min_{\epsilon \geq 0} \frac{\epsilon}{2} \text{ s.t. } \begin{cases} |d_X(x_i, x_j) - d_Y(y'_i, y'_j)| \leq \epsilon \\ |d_Y(y_k, y_l) - d_X(x'_k, x'_l)| \leq \epsilon \\ |d_X(x_i, x'_k) - d_Y(y_k, y'_i)| \leq \epsilon. \end{cases} \]

*Bronstein, Bronstein & Kimmel, 2006*
Numerical example

Canonical forms
(MDS based on 500 points)

Gromov-Hausdorff distance
(GMDS based on 50 points)

Bronstein, Bronstein & Kimmel, 2006
Extrinsic similarity using Gromov-Hausdorff distance

**EXTRINSIC SIMILARITY**

Congruence

\[ \exists i \in \text{Iso}(\mathbb{E}) \text{ s.t. } i(Y) = X \]

ICP distance:

\[ d_{\text{ICP}}(X, Y) = \min_{i \in \text{Iso}(\mathbb{E})} d_{\text{H}}(X, i(Y)) \]

Euclidean isometry

\[ (X, d_{\mathbb{E}}) \sim (Y, d_{\mathbb{E}}) \]

GH distance with Euclidean metric:

\[ d_{\text{GH}}((X, d_{\mathbb{E}}), (Y, d_{\mathbb{E}})) \]

Connection between Euclidean GH and ICP distances:

\[ d_{\text{GH}}((X, d_{\mathbb{E}}), (Y, d_{\mathbb{E}})) \leq d_{\text{ICP}}(X, Y) \leq c \sqrt{d_{\text{GH}}((X, d_{\mathbb{E}}), (Y, d_{\mathbb{E}}))} \]

Mémoli (2008)
Connection to canonical form distance

\[ d_{GH}(X,f(X)) \leq \delta_X \]

\[ d_{ICP}(f(X),g(Y)) \]

\[ d_{GH}(Y,g(Y)) \leq \delta_Y \]
Correspondence again

A different representation for correspondence using indicator functions

\[ w(x, y) = \begin{cases} 
1 & (x, y) \in C \\
0 & \text{else.} 
\end{cases} \]

\( w : X \times Y \rightarrow \{0, 1\} \) defines a valid correspondence if

\[ \int_X w(x, y) \, dx > 0 \]

\[ \int_Y w(x, y) \, dy > 0 \quad \forall x, y \]
L_p Gromov-Hausdorff distance

We can give an alternative formulation of the Gromov-Hausdorff distance

\[
d_{GH}(X, Y) = \frac{1}{2} \inf_{C} \sup_{x, y \in C} |d_X(x, x') - d_Y(y, y')|
\]

\[
= \frac{1}{2} \inf_{w} \sup_{x, x' \in X} \int_{y, y' \in Y} |d_X(x, x') - d_Y(y, y')| w(x, y) w(x', y') dx dy > 0
\]

s.t. \[
\int_{X} w(x, y) dx > 0
\]
\[
\int_{Y} w(x, y) dy > 0 \quad \forall x, y
\]

Can we define an L_p version of the Gromov-Hausdorff distance by relaxing the above definition?
Measure coupling

Let $\mu_X, \mu_Y$ be probability measures defined on $X$ and $Y$ (a metric space with measure is called a metric measure or mm space).

A measure $\mu$ on $X \times Y$ is a coupling of $\mu_X$ and $\mu_Y$ if

$$\mu(X' \times Y) = \mu_X(X')$$
$$\mu(X \times Y') = \mu_Y(Y')$$

for all measurable sets $X' \subseteq X, Y' \subseteq Y$.

The measure $\mu$ can be considered as a relaxed version of the indicator function $\psi$ or as fuzzy correspondence.

Mémoli, 2007
Gromov-Wasserstein distance

The relaxed version of the Gromov-Hausdorff distance is given by

\[ d^p_{GH}(X, Y) = \frac{1}{2} \inf \mu \left( \int_{X \times X} \int_{Y \times Y} |d_X(x, x') - d_Y(y, y')|^p d\mu(x, y) d\mu(x', y') \right)^{1/p} \]

and is referred to as Gromov-Wasserstein distance [Memoli 2007]
Earth Mover’s distance

Let \((\mathcal{Z}, d_{\mathcal{Z}})\) be a metric space, \(X, Y \subseteq \mathcal{Z}\) and \(\mu_X, \mu_Y\) measures supported on \(X, Y\).

Define the coupling \(\mu\) of \(\mu_X, \mu_Y\).

The *Wasserstein* or *Earth Mover’s distance* (EMD) is given by

\[
d_{\text{EMD}}^p(X, Y) = \inf_{\mu} \left( \int_{X \times Y} d_{\mathcal{Z}}^p(x, y) \, d\mu(x, y) \right)^{1/p}
\]

**EMD as optimal mass transport:**

- \(\mu(x, y)\) mass transported from \(x\) to \(y\)
- \(d_{\mathcal{Z}}(x, y)\) distance traveled

Mémoli, 2007
The analogy

<table>
<thead>
<tr>
<th><strong>Hausdorff</strong></th>
<th><strong>Wasserstein</strong></th>
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<tbody>
<tr>
<td>Distance between <strong>subsets</strong> $(X, d_X), (Y, d_Y)$ of a metric space $(\mathbb{Z}, d_{\mathbb{Z}})$.</td>
<td>Distance between subsets $X, Y$ of a metric measure space $(\mathbb{Z}, d_{\mathbb{Z}}, \mu)$.</td>
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GMDS with geodesic distances
GMDS with diffusion distances

BBK, M. Mahmoudi, G. Sapiro, 2009